Theoretical and numerical analysis of a coupled system of second order non-linear differential equations

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Summary. This paper deals with a coupled system of non-linear elliptic differential equations arising in electrodeposition modelling process. We show the existence and uniqueness of the solution. A numerical algorithm to compute an approximation of the weak solution is described. We introduce a domain decomposition method to take in account the anisotropy of the solution. We show the domain decomposition method convergence. A numerical example is presented and commented.

1 Introduction

Electrodeposition of alloys based on the iron group of metals is one of the most important recent developments in the field of alloy deposition. In [KAP97], [SP98] Pritzker et al have proposed a model which involves the one-dimensional steady-state transport of the various species with simultaneous homogeneous reactions. The concentration of different species that are involved satisfies a system of non-linear differential equations. In this paper we are concerned with a reduced problem arising in one step of an iterative method solving the whole system. More precisely we consider the following system:

$$\begin{cases} -dv'' + b(x)v' - m(v\Phi')' = f & \text{in } (0,\delta) \\ v(\delta) = v^* & \\ -dv'(0) - m\,v(0)\Phi'(0) = -\gamma\, v(0) & \\ -[p(v)\Phi']' = q(v) & \text{in } (0,\delta) \\ \Phi(0) = V_0, \Phi(\delta) = 0, & \end{cases} \tag{1}$$

where v is the concentration, Φ is the potential, f denotes the production rate, d is the diffusion coefficient, m is the electrical mobility, δ is a fixed

nonnegative real, v^*, V_0, γ are constants, p, q are nonnegative functions and $b(x) = -ax^2$ is the fluid velocity vector, with a a nonnegative constant.
In section 2 we give a proof of existence and uniqueness of the solution (v, Φ) of system (1) in $C^2([0,\delta]) \times C^2([0,\delta])$.
The numerical solution of the system considered in the electrodeposition are characterized by stiff variations near the boundary $x = 0$. In order to take account of the anisotropy of the solution we introduce in section 3 a generalized version of the two domain decomposition method due to F. Gastaldi, L. Gastaldi and A. Quarteroni (see [GGQ96]). We give a sketch of the proof for the convergence in the new case of non constant coefficients and Robin boundary conditions in $x = 0$. In section 4 we present and discuss the result of a numerical example.

2 Existence and Uniqueness Result

Let $\varepsilon > 0$. We introduce the following assumptions:
H01) $p \in C^1(\mathbb{R})$ and there exist nonnegative constants η_0 and η_1 such that : $\eta_0 \le p \le \eta_1$.
Let $k_1 > 0$ such that $\mid p(x) - p(y) \mid \le k_1 \mid x - y \mid \quad \forall x, y \in [0, v^* + \varepsilon]$.
H02) There exist two nonnegative constants k_2 and η_2 such that $-\eta_2 \le q \le \eta_2$ and $\mid q(x) - q(y) \mid \le k_2 \mid x - y \mid \quad \forall x, y \in [0, v^* + \varepsilon]$.
H03) The constant d is such that:
i) $d > \gamma\delta + \dfrac{2a\delta^3}{3} + \dfrac{m(V_0 + 2\eta_2\delta^2)((v^* + \varepsilon)k_1 + \eta_1)}{\eta_0^2} + \dfrac{2m(v^* + \varepsilon)k_2\delta^2}{\eta_0}$.
ii) $d \ge \dfrac{1}{\min(v^*, \varepsilon)}\{\parallel f \parallel \delta^2 + (\gamma\delta + \dfrac{2a\delta^3}{3} + \dfrac{mV_0}{\eta_0} + \dfrac{2m\eta_2\delta^2}{\eta_0})(v^* + \varepsilon)\}$.

Theorem 1. *Under assumptions H01-H03 the system (1) has a unique solution $(v, \Phi) \in C^2([0,\delta]) \times C^2([0,\delta])$.*

Proof. Let Π the map defined from $C([0,\delta])$ to $C([0,\delta])$ by $\Pi v = u$, where for $x \in [0,\delta]$

$$\begin{aligned} u(x) = v^* + \frac{\gamma}{d}(x-\delta)v(0) + \frac{1}{d}\int_\delta^x [(bv)(y) - \int_0^y (b^{'}v)(t)dt - \int_0^y f(t)dt]dy \\ -\frac{m}{d}\int_\delta^x \{\frac{v(y)}{p(v)(y)}[-\frac{V_0}{\delta} + \frac{1}{\delta}\int_0^\delta\int_0^t q(v)(s)dsdt - \int_0^y q(v)(t)dt]\}dy. \end{aligned} \tag{2}$$

By integration of (1) it follows that a solution of the system is a fixed point of application Π. We set $D = \{u \in C([0,\delta]), 0 \le v \le v^* + \varepsilon\}$ equipped with the uniform norm. Using hypotheses H01-H03 we prove that the map Π is a contraction from D into itself. By Schauder fixed point theorem it comes that

Π has a unique fixed point $v \in D$ and by (2) $v \in C^2([0,\delta])$. Then (1) has a unique solution $(v,\Phi) \in C^2([0,\delta]) \times C^2([0,\delta])$. With

$$\Phi(x) = -\int_{\delta}^{x} \{ \frac{1}{p(v)(s)} (\frac{V_0}{\delta} + \frac{1}{\delta} \int_0^{\delta} \int_0^{y} q(v)(t)dtdy - \int_0^{s} q(v)(y)dy) \} ds. \quad (3)$$

3 Numerical Methods

For convenience we introduce the following new unknowns:

$$\psi(x) = \Phi(x) - \frac{V_0}{\delta}(\delta - x) \text{ and } w(x) = v(x) - v^* \text{ for all } x \in [0,\delta]. \quad (4)$$

System (1) is then equivalent to the following systems:

$$\begin{cases} L_1 w = F(w,\psi) \quad \text{in } (0,\delta), \\ w(\delta) = 0,\ -dw'(0) = G(w,\psi)(0). \end{cases} \quad (5)$$

and

$$\begin{cases} -[p(w+v^*)\psi']' = q(w+v^*) \text{ in } (0,\delta) \\ \psi(0) = 0,\ \psi(\delta) = 0, \end{cases} \quad (6)$$

where:

$$\begin{cases} L_1 w = -dw'' + B_0(x)w',\ B_0(x) = b(x) + m\frac{V_0}{\delta} \quad x \in (0,\delta). \\ F(w,\psi) = m[(w+v^*)\psi']' + f \quad x \in (0,\delta). \\ G(w,\psi) = [m(\psi'(0) - \frac{V_0}{\delta}) - \gamma](w(0)+v^*) \quad x \in (0,\delta). \end{cases} \quad (7)$$

The iterative method considered to solve this coupled problem first solves the equation (5) for a given potential ψ_n and then using the same algorithm solves equation (6) for a given concentration w_n.

Let w_0 be the solution of (5) with $F = 0$ and then for any $n \in N$, w_{n+1} is the solution of the linear system:

$$\begin{cases} L_1 w = F(w_n,\psi) \qquad \text{in } (0,\delta), \\ w(\delta) = 0,\ -dw'(0) = G(w_n,\psi)(0). \end{cases} \quad (8)$$

The existence and uniqueness of a solution of problem (8) is trivial in $C^2([0,\delta])$.

3.1 Iterative Method to Solve the Equation (8)

Let $c \in (0,\delta)$ be fixed. To solve equation (8) using the iterative domain decomposition method we decompose the set $(0,\delta)$ in two non-overlapping subdomains, $\Omega_1 = (0,c)$ and $\Omega_2 = (c,\delta)$. In the subdomain Ω_1 we consider a finer mesh structure than in Ω_2.

Let $n \in N$, A and B two reals parameters such that $AB \leq 0$, $A \neq B$.

Given $w_{1,0} = w_{2,0} = w_n$ and $\lambda^0 = d(w_{2,0})'(c) - (\frac{1}{2}B_0(c) + A)w_{2,0}(c)$, for each $k \geq 0$ we have to solve

$$\begin{cases} L_1 w_{1,k+1} = F(w_n, \psi) \text{ in } H^1(0,c), \\ -d(w_{1,k+1})'(0) = G(w_n, \psi)(0), \\ d(w_{1,k+1})'(c) - (\frac{1}{2}B_0(c) + A)w_{1,k+1}(c) = \lambda^k, \end{cases} \tag{9}$$

and then

$$\begin{cases} L_1 w_{2,k+1} = F(w_n, \psi) \text{ in } H^1(c,\delta), \\ w_{2,k+1}(\delta) = 0, \\ d(w_{2,k+1})'(c) - (\frac{1}{2}B_0(c) + B)w_{2,k+1}(c) = \\ d(w_{1,k+1})'(c) - (\frac{1}{2}B_0(c) + B)w_{1,k+1}(c), \end{cases} \tag{10}$$

with

$$\lambda^{k+1} = d(w_{2,k+1})'(c) - (\frac{1}{2}B_0(c) + A)w_{2,k+1}(c). \tag{11}$$

Thanks to the Lax-Milgram Theorem we can able to prove the:

Proposition 1. *If $A \leq 0$, then the problem (9) has a unique solution $w_{1,k+1} \in C^2([0,c])$ and if $B \geq 0$, then the problem (10) has a unique solution $w_{2,k+1} \in C^2([c,\delta])$.*

We will now give a sketch of the proof of convergence of the subdomain decomposition algorithm (9) and (10) applied to the solution of the linear problem (8) taking in account an anisotropic advective field and non constant absorption terms.

Proposition 2. *Let $c \in (0,\delta)$ such that $2d >\mid B_0(c) + A + B \mid$. Then the sequence $(w_{1,k}, w_{2,k})$ converge to (v,v) in $C(0,c) \times C(0,c)$.*

Proof. Let us define the errors $e_{j,k} = v - w_{j,k}$; $j = 1,2$, and study their behavior as k grows. We can prove the following inequality:

$$\| e_{1,k+1} \|_\infty \leq \gamma_0 \| e_{1,k} \|_\infty \ \textit{and} \ \| e_{2,k+1} \|_\infty \leq \gamma_0 \| e_{2,k} \|_\infty, \tag{12}$$

with $\gamma_0 > 0$. Conditions $A < B$ and $2d >\mid B_0(c) + A + B \mid$ imply that $\gamma_0^2 < 1$ which finish the proof.

4 Numerical Result

The algorithm introduced in the previous section has been implemented numerically for one example of problem (1) with $\delta = 11341 * 10^{-9}$, $c = \delta/10$, $m = 52133 * 10^{-12}$, $d = 68 * 10^{-11}$, $\gamma = 0.05$, $v^* = 1$, $V_0 = -0.85$ and

$a = 660.45$, $p = 1 + \dfrac{1}{x^2+x+1}$, $q = \dfrac{1}{|\,x\,|+1}$ and $\dfrac{10\delta}{m+x}$.
This is a nonlinear system with nondifferentiable second member. The numerical concentration was plotted in figure 1.
We remark that the variation rate of v is very strong near the boundary 0.

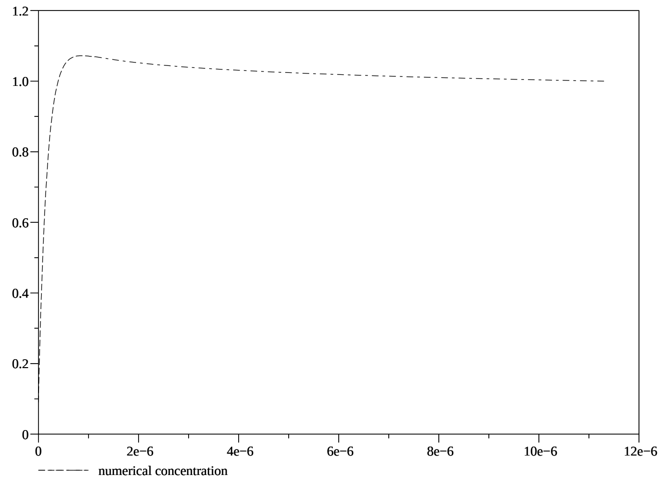

Fig. 1. Numerical concentration solution for $f = 10 * \delta/(m + x)$

This property justifies the use of the domain decomposition method and the choice of the fictitious boundary c near 0.
The algorithm (9)-(10) converges with $N_1 = 70$ finite element at the sub-domain $[0, c]$ and $N_2 = 50$ finite element at the sub-domain $[c, \delta]$. We stop when the error is of order 10^{-19}.

References

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